5. CONCLUSIONS

We have discussed a metric which allows a four-parameter isometry group G_4IV. When this metric is used in Einstein's field equations with a dust energy–momentum tensor, one finds solutions only for a tachyon dust. The solutions are static and have singularities on timelike hypersurfaces. Our solutions do not, of course, prove anything concerning the existence or nonexistence of tachyon dust in nature. What they do show is that the Einstein field equations have solutions for such a hypothetical dust. In fact the metric we considered has solutions only for this type of dust.

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1 $i, j, k \cdots$ are tensor indices which take on the values 1, 2, 3, 4. $a, b, c, \cdots$ label the Killing vectors, infinitesimal operators, etc. For a four-parametric group these also take on the values 1, 2, 3, 4. We choose the signature (+, +, +, −). Also we use units with $c = 1$, $8\pi k = 1$, where c is the speed of light in vacuum and k is the Newtonian gravitational constant.

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3 See Ref. 2, p. 229. Our case is classified as G_4IV in Ref. 2.
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Time-Dependent and Time-Independent Potential Scattering for Asymptotically Coulomb Potentials

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We consider the three-dimensional quantum mechanical problem of nonrelativistic potential scattering, where the Hamiltonian is $H_2 = H_1 + V = H_0 + V_c + V$; H_0 is the free particle Hamiltonian, V_c is the Coulomb potential, and $V(x)$ is a real-valued potential function defined for $x \in R^3$. We show the existence of the wave operators $W_\pm(H_2, H_D) = \text{s-lim}\, e^{iH_2 t} e^{-iH_D(t)}$ on $L^2(R^3)$ for $V = V_2 + V'$, where $V_2 \in L^2(R^3)$, $V' \in L^\infty(R^3) \cap L^p(R^3)$ ($p < 3$), and $e^{-iH_D(t)}$ is the family of unitary operators used by Dollard to show the existence of the wave operators $W_\pm(H_1, H_D)$ appropriate for the pure Coulomb case. If V is spherically symmetric then $W_\pm(H_2, H_D)$ are shown to be absolutely continuous complete. If in addition $V(r)$ is continuous in $(0, \infty)$, then $W_\pm(H_2, H_D)$ are continuous complete. In both cases $S = W_+^* W_-$ is unitary. The connection between the more physical, time-dependent wave operator approach and the traditional time-independent method is made, and phase shift formulas are obtained for the wave and scattering operators.

INTRODUCTION

One of the first problems to be explicitly solved in nonrelativistic quantum mechanics was the Coulomb scattering problem by the time-independent eigenfunction expansion method.[1] Scattering solutions are obtained by imposing certain asymptotic conditions on the solutions $\Phi_1^\pm(x, q)$ of the partial differential equation $(-\Delta + V_c - k^2)\Phi_1^\pm = 0$ (see Appendix A). Throughout this paper, x and q will be used to denote elements of R^3 while r and k will denote their magnitudes.

Let us consider the three-dimensional nonrelativistic system described by the Hamiltonian $H_2 = H_0 + V_c + V = H_1 + V$, where H_0 is the free particle Hamiltonian, V_c is the Coulomb potential, and V is a real-valued potential function. H_i ($i = 0, 1, 2$) act in the Hilbert space $\mathcal{H} = L^2(R^3)$. Recently the existence of certain modified wave operators

$$W_\pm(H_2, H_D) = \underset{t \to \pm\infty}{\text{s-lim}}\, e^{iH_2 t} e^{-iH_D(t)}$$

appropriate for systems where the potential function is asymptotically Coulomb was shown by Dollard,[2] thus establishing a more physical time-dependent description of the scattering process. Furthermore, when the potential is pure Coulomb, Dollard[2] showed the absolutely continuous spectrum completeness (also continuous spectrum completeness) of the wave operators $W_\pm(H_1, H_D)$ and established the relation between the customary scattering solutions of the time-independent theory. Support for the acceptance of the wave operators $W_\pm(H_1, H_D)$ as the appropriate ones for the description of the pure Coulomb scattering process is given by the following facts[2,3]:

(1) Let P_1 be the projection on the continuous spectrum subspace of H_1. For any vector $f \in P_1\mathcal{H}$ there exists vectors $g_\pm \in \mathcal{H}$ such that

$$\lim_{t \to \pm\infty} \| e^{-iH_1 t} f_\pm - e^{-iH_D(t)} g_\pm \| = 0.$$

Conversely, for every $g \in \mathcal{H}$ there exist vectors $f_\pm \in \mathcal{H}$ such that

$$\lim_{t \to \pm\infty} \| e^{-iH_1 t} f_\pm - e^{-iH_D(t)} g \| = 0.$$

This implies the existence of the wave operators $W_\pm(H_1, H_D)$.

Furthermore

$$\lim_{t \to \pm\infty} \int_B | e^{-iH_1 t} f(x) |^2 dx = 0$$

for any compact subset B of R^3, so that indeed $e^{-H_1 t} f$ behaves as a traveling wavepacket.

(2) The momentum and position probability distributions of the vector $e^{-iH_D(t)} g_+$ ($e^{-iH_D(t)} g_-$) in the $t \to +\infty$ ($t \to -\infty$) limit are the same as those of the vector $e^{-iH_0 t} g_+$ ($e^{-iH_0 t} g_-$).

The momentum distribution of $e^{-iH_0 t} h$ is just $|h_0(k)|^2$ and the position probability distribution is given by $(m/t)^3 |h_0(mx/t)|^2$ as t approaches $\pm\infty$ (h_0 is the three-dimensional Fourier transform of h).

(3) $(W_\pm g)(x) = \int \Phi_1^\pm(x,q) g_0(q)dq = f_\pm(x)$ and the time evolution $e^{-iH_1t}f_\pm$ is given by $f_\pm(x,t) = \int \Phi_1^\pm(x,q) \times e^{-ik^2t} g_0(q)dq$ thus allowing the identification of the variable q occurring in $\Phi_1^\pm(x,q)$ as the asymptotic $(t \to \pm\infty)$ momentum index.

In this article we obtain results similar to those of Dollard for a system described by the Hamiltonian H_2. In Sec. 2 we extend Dollard's existence proof for $W_\pm(H_2, H_D)$ to include a larger class of potentials (not necessarily spherically symmetric), which include potentials behaving as $r^{-1-\epsilon}$ as $r \to \infty$ $(\epsilon > 0)$. In the case when V is spherically symmetric, we show absolutely continuous completeness of the wave operators $W_\pm(H_2, H_D)$. Since an eigenfunction expansion has not been shown to exist for H_2 for nonspherically symmetric potentials, we use Kodaira's theory of eigenfunction expansions[4] for spherically symmetric potentials and consider direct sums over the angular momentum to establish the relation between the wave operators $W_\pm(H_2, H_D)$ and the time-independent method.

The notation we will use is introduced in Sec. 1. Some facts from the theory of eigenfunction expansion are reviewed there also. In Sec. 2 the theorems are presented: Their proofs follow in Sec. 3. Appendix A gives formulas connecting three- and one-dimensional expressions. In Appendix B we give a heuristic argument, for spherically symmetric V, which indicates that the wave operators $W_\pm(H_2, H_D)$ exist and are absolutely continuous complete when V satisfies Kodaira's criterion [see Eqs. (1a)—(1c)] for the existence of an eigenfunction expansion for H_2.

For another approach to this problem the reader is referred to the article by Mulherin and Zinnes.[5] For the treatment of the existence and absolutely continuous completeness question, without the spherically symmetric restriction, see forthcoming work of B. Simon. For the case of systems described by potentials of the type c/r^α $(0 < \alpha < 1)$ see Amrein, Martin, and Misra.[6] We also mention that results analogous to ours have been obtained by Green and Lanford[7] (see also Kuroda[8]) for the case $V_c = 0$, $H_2 = H_0 + V$.

1. PRELIMINARIES

Throughout this paper x, q denote elements of $\mathbb{R}^3$ while r, k will denote their magnitudes. We adhere to the Hilbert space terminology of Kato.[9] We consider the formally symmetric operators H_i $(i = 0,1,2)$, where H_0 is the kinetic energy operator, H_1 is $H_0 + V_c$ $(V_c = e_1e_2/r$, the Coulomb potential), and $H_2 = H_1 + V$. V is a real-valued measurable function defined for $x \in \mathbb{R}^3$. The operators H_i act in $\mathcal{L}^2(\mathbb{R}^3)$ and their definition as self-adjoint operators will be obtained either from a form-bounded or operator relative-boundedness condition. We will not change notation for the corresponding self-adjoint extensions of these operators. When we are dealing with a spherically symmetric potential V we can also consider these operators as direct sums, i.e., $H_i = \sum_{l=0}^\infty \oplus H_i^l (i = 0,1,2)$, which act in the direct sum of the Hilbert spaces $\mathcal{H}^l$. $\mathcal{H} = \sum_{l=0}^\infty \oplus \mathcal{H}^l$ is isomorphic to $\mathcal{L}^2(\mathbb{R}^3)$ (see Green and Lanford[7]). The subspace $\mathcal{H}^l$ is the Hilbert space $\mathcal{H}^l = \mathcal{L}^2(0,\infty) \otimes \mathcal{L}_l^2(\Omega)$, where $\mathcal{L}_l^2(\Omega)$ is the $(2l+1)$-dimensional Hilbert space with orthonormal basis elements given by the spherical harmonics $Y_{lm}(\Omega)$, $-l \le m \le l$, and $\mathcal{L}^2(0,\infty)$ is the radial Hilbert space with measure dr. H_i^l $(i = 0,1,2)$ acts trivially on the second factor. The definition of H_i^l $(i = 0,1,2)$ as a self-adjoint operator in $\mathcal{L}^2(0,\infty)$ will be taken from the theory of eigenfunction expansions as developed by Kodaira[4] which we will briefly review. These operators are self-adjoint restrictions of the differential operators

$$\mathcal{L}_i^l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V_i'(r), \quad i = 0,1,2,$$

where $V_0' = 0$, $V_1' = V_c$, and $V_2' = V_c = V_c + V$. For $l = 0$ we are in the limit circle case so we impose the boundary condition $u(0) = 0$ for $u \in D(H_i^0)$. For the precise specification of the domains of these operators see Kodaira[4] or Stone.[10] When V satisfies the conditions

$V(r)$ continuous on each closed subinterval of $(0,\infty)$ (1.1a)

$$V(r) \sim O(1/r^{2-\epsilon}) \quad \text{for } r \to 0,\ \epsilon > 0, \tag{1.1b}$$

$$V(r) \sim O(1/r^{1+\epsilon'}) \quad \text{for } r \to \infty,\ \epsilon' > 0, \tag{1.1c}$$

then H_i^l $(i = 0,1,2)$ admit eigenfunction expansions, i.e., for any $f \in \mathcal{L}^2(0,\infty)$ (we suppress the l dependence),

$$f(r) = \text{l.i.m.} \sum_{n=1}^N u_{1n}(r)(u_{1n}, f) + \underset{N\to\infty}{\text{l.i.m.}} \int_{N^{-1}}^N u_i(r,k) \int_0^\infty u_i(r,k) f(r)dr\ dk,$$

where u_{1n} are the orthonormal eigenvectors of the point spectrum of H_i^l and $u_i(r,k)$ is a real continuous function of r $(0 \le r < \infty)$ and k $(k > 0)$, which satisfies

$$(L_i^l - k^2)u_i^l(r,k) = 0, \quad k > 0, \qquad u_i^l(r,k) \to C(k) r^{l+1}, \quad \text{as } r \to 0,\ k > 0, \tag{1.2a}$$

$$u_0(r,k) = (2/\pi)^{1/2} kr j_l(kr) \to (2/\pi)^{1/2} \sin(kr - l\pi/2) \quad \text{as } r \to \infty,\ k > 0, \tag{1.2b}$$

$$u_i(r,k) \to (2/\pi)^{1/2} \sin(kr - (\alpha/k)\log(2kr) - l\pi/2 + \delta_i^l) \quad \text{as } r \to \infty,\ i = 1,2,\ k > 0$$

and $\delta_1^l = \arg[\Gamma(l + 1 + i\alpha/k)]$, $\alpha = e_1e_2m$. The position of the continuous spectrum of H_i for all i and l is $[0,\infty)$ and is absolutely continuous.

We denote the transform

$$f_i(k) = (F_if)(k) = \text{l.i.m.} \int_0^\infty u_i(r,k)f(r)dr, \quad f \in \mathcal{L}^2(0,\infty) \tag{1.3}$$

$$(F_i^* g)(r) = \text{l.i.m.} \int_0^\infty u_i(r,k)g(k)dk, \quad g \in \mathcal{L}^2(0,\infty).$$

Note that F_i is partially isometric and

$$F_i^* F_i = P_i, \quad F_i F_i^* = P_i', \tag{1.4}$$

where P_i is the projection from $\mathcal{L}^2(0,\infty)$ onto the continuous spectrum (which is absolutely continuous) subspace of H_i and P_i' is the projection from $\mathcal{L}^2(0,\infty)$ onto $F_i(\mathcal{L}^2(0,\infty))$. Formulas relating one-dimensional and three-dimensional quantities will be found in Appendix A.

We will also consider various generalized wave operators acting in $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^3)$. We let

$$W_\pm(H_j, H_i) = \text{s-}\lim_{t\to\pm\infty} e^{iH_j t} e^{-iH_i t} P_i, \quad i,j = 1,2, \quad (1.5a)$$

where H_i, H_j are self-adjoint operators and P_i is the projection operator on the subspace of absolute continuity of H_i. We also will use the modified wave operators introduced by Dollard[2] which we denote by

$$W_\pm(H_1, H_D) = \text{s-}\lim_{t\to\pm\infty} e^{iH_i t} e^{-iH_D(t)}, \quad i = 1,2, \quad (1.5b)$$

where $e^{-iH_D(t)}$ is unitary (but not a one-parameter group) and

$$H_D(t) = H_0 t + \epsilon(t) 1/2 e_1 e_2 (H_0)^{-1} \log(-4|t|H_0), \quad (1.6)$$

where $\epsilon(t) = \pm 1$ for $t \gtrless 0$. When the wave operators $W_\pm(H_j, H_f)$, $j = 1,2$, $f = 1,2,D$ exist and $R(W_\pm)$ are the absolutely continuous (continuous) spectrum subspace of H_j, then $W_\pm(H_j, H_f)$ will be said to be absolutely continuous (continuous) complete, abbreviated acc (cc). When considering these operators acting on $\mathcal{H}^l = \sum_{l=0}^\infty \oplus \mathcal{H}^l$ they will be denoted by $W_\pm = \sum_{l=0}^\infty \oplus W_\pm^l$. This notation is justified as a result of Kuroda[8] which shows that $W_\pm$ exist if and only if $W_\pm^l$ exist for each l and when $W_\pm$ exist, $W_\pm = \sum_{l=0}^\infty \oplus W_\pm^l$.

2. RESULTS

In part A of this section we give results on the existence and completeness of wave operators deduced from the time-dependent wave operator approach. In part B we give time-independent results and the relation between the time-dependent and time-independent results.

A. Time-Dependent Scattering

Contained in Dollard's work is the following:

Theorem 2.1[2]*:* In $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^3)$ we have that

(a) The wave operators $W_\pm(H_1, H_D)$ exist and are acc and cc.

(b) For $V \in \mathcal{L}^2(\mathbb{R}^3)$ and real, $W_\pm(H_2, H_D)$ exist.

We give the following extension of Theorem 2.1.

Theorem 2.2: Let $V = V_2 + V'$ with $V_2 \in \mathcal{L}^2(\mathbb{R}^3)$ and $V' \in \mathcal{L}^\infty(\mathbb{R}^3) \cap \mathcal{L}^p(\mathbb{R}^3)$ $(p < 3)$. Then the wave operators $W_\pm(H_2, H_D)$ exist.

With H_0, H_1, H_2 defined as direct sums over l and H_1^l, H_2^l defined as operator sums with $\mathfrak{D}(H_i^l) = \mathfrak{D}(H_0^l)$, $i = 1,2$, we state a previously obtained result as the following theorem.

Theorem 2.3[11]*:* If V is spherically summetric and satisfies the condition

$$\int_0^R r^2 |V(r)|^2 dr + \int_R^\infty (1+r)^\delta |V(r)|^i dr < \infty, \quad i = 1,2, \quad (2.1)$$

for some R $(0 \le R < \infty)$ and some δ $(0 < \delta < 1)$, then the generalized wave operators $W_\pm(H_2, H_1)$ and $W_\pm(H_1, H_2)$ exist on $\mathcal{L}^2(\mathbb{R}^3)$ and are acc. Furthermore, if $V(r)$ also satisfies Eqs. (1a)–(1c), the above wave operators are cc.

Remark 2.1: In proving the first part of the above theorem we applied a theorem of Kuroda[12] for operators. Kuroda[13] has an analogous theorem for forms which he used to extend Green and Lanford's results[7] for the case of the wave operators $W_\pm(H_2, H_1)$ with $V_c = 0$. We have not been able to obtain the required estimates to apply this theorem which would allow a $r^{-2+\epsilon}$ $(\epsilon > 0)$ behavior for V at the origin. In Appendix B we give an argument which indicates but does not prove existence and absolutely continuous completeness in this case.

Remark 2.2: With the condition of Eqs. (1a)–(1c) imposed on V the continuous spectrum of H_2 is absolutely continuous from Kodaira's theory of eigenfunction expansions.[4] Precisely to what extent the continuity assumption of V can be relaxed and still have cc has not been investigated.

Combining a modified version of the chain rule for generalized wave operators and Theorem 2.1, we have the following:

Theorem 2.4:

(a) Let V satisfy the conditions of (2.1). Then the wave operators $W_\pm(H_2, H_D)$ exist and are acc.

(b) If V satisfies both (2.1) and (1a)–(1c) then $W_\pm(H_2, H_D)$ exist and are cc.

(c) In both (a) and (b) the scattering operator $S = W_+^* W_-$ is unitary.

B. Time-Independent Scattering and the Relation to Time-Dependent Scattering

We give these results in one-dimensional form and suppress the l dependence. The three-dimensional results are obtained by taking direct sums. Dollard's results on cc for the wave operators $W_\pm(H_1, H_D)$ can then be stated as Theorem 2.5.

Theorem 2.5[2]*:* Defining

$$(U_\pm(H_1, H_D) f)(r) = \text{l.i.m.} \int u_1(r,k) e^{\mp i\delta_1(k)} (F_0 f)(k) dk$$

(l.i.m. with respect to the k integral means norm limit over a sequence $[\alpha, \beta]$ of k intervals, $\alpha > 0$, $\beta < \infty$, with $\alpha \to 0$ and $\beta \to \infty$) or

$$U_\pm(H_1, H_D) f = F_1^* e^{\mp i\delta_1} F_0 f, \quad (2.2)$$

then $U_\pm(H_1, H_D) = W_\pm(H_1, H_D)$.

We are now in a position to establish the time-independent theory for asymptotically Coulomb potentials and the relationship to the time-dependent theory. We have the following theorem.

Theorem 2.6:

(a) Let V satisfy (1.1a)–(1.1c) and define $S' = F_0^* e^{2i\delta_2} F_0$; then S' is unitary.

(b) Let V satisfy (1a)–(1c) and (2.1). Define $U_\pm(H_2, H_1) = F_2^* e^{\pm i(\delta_1 - \delta_2)} F_1$ on $\mathcal{L}^2(0,\infty)$; then $U_\pm(H_2, H_1) = W_\pm(H_2, H_1)$.

(c) With V as in part b we have $W_\pm(H_2, H_D) = F_2^* e^{\mp i\delta_2} F_0$. Define $S = W_+^*(H_2, H_D) W_-(H_2, H_D)$; Then $S = S'$.

Remark 2.3: Knowing the eigenfunction expansion for H_2, such as having δ_2, we can always define a unitary operator by $F_0^* e^{2i\delta_2} F_0$. The important point is that the time-independent definition agrees with the operator S defined from the time-dependent theory.

Remark 2.4: We expect the equality $W_\pm(H_2, H_1) = U_\pm(H_2, H_1)$ to be valid under the less restrictive conditions on V of (1a)-(1c) or more generally when an eigenfunction expansion is valid for H_2 and the continuous spectrum is absolutely continuous.

3. PROOF OF THE THEOREMS

For the proof of Theorems 2.1 and 2.5 see Dollard.[2] For the proof of Theorem 2.3 see O'Carroll.[11]

Proof of Theorem 2.2: We apply Theorem 3.7, p. 533 of Kato.[9] For a fundamental subset in $L^2(R^3)$ we take the space of functions $\mathcal{C} \subset S(R^3)$ used by Dollard,[2] where $f(x) \in \mathcal{C}$ if $f_0(q)$ vanishes in a neighborhood of $q = 0$. For $h \in \mathcal{C}$ we show that

$$\|(V_2 + V')e^{-iH_D(t)}h\|_2 \le \|V_2 e^{-iH_D(t)}h\|_2 + \|V' e^{-iH_D(t)}h\|_2 \quad (3.1)$$

is integrable on $[t_0, \infty]$ where $t_0 > 1$. We use Lemma 2 of Dollard[2]: Let $h \in \mathcal{C}$. Then, in the system of units used in the introduction for $|t| \ge t_0$, we have

$$(e^{-iH_D(t)}h)(x) = (1/2\mathrm{it})^{3/2}\phi_c(x)h_0(x/2t) + (\tfrac{1}{4}\pi i t)^{3/2} e^{ix^2/4t} R_h(x,t), \quad (3.2)$$

where

$$\phi_c(x) = \exp(ix^2/4t)\exp\{-i\epsilon(t)e_1e_2\log(x^2/|t|)t/2x\}$$

and for any integer n there exists a number $\mu \ge 0$ and a constant K, depending on h, such that

$$R_h(x,t) \le K(\log|t|)^\mu/|t|^{1/2}[1 + (x/t)^2]^n. \quad (3.3)$$

Returning to (3.1) we find

$$\|V_2 e^{-iH_D(t)}h\|_2 \le \|V_2\|_2 \|e^{-iH_D(t)}h\|_\infty \le \|V_2\|_2\{(2t)^{-3/2}\|h\|_1 + K(\log|t|)^\mu t^{-2}\} \quad (3.4)$$

which is integrable on $[t_0, \infty)$. The second term on the right side of (3.1) is found to satisfy

$$\begin{aligned}
&\|V' e^{-iH_D(t)}h\|_2 \\
&\le \|V'\phi_c(x)(2t)^{-3/2}h_0(x/2t)\|_2 + \||V'|(2t)^{-3/2}|R_h|\|_2 \\
&= \|V'^2(2t)^{-3}|R_h|^2\|_1^{1/2} + I' \\
&\le \|V'^2\|_p^{1/2} t^{-3/2}(\log|t|)^\mu t^{-1/2} t^{3/2p'} I^{1/2p'} + I' \\
&\le K''\{(\log|t|)^\mu/t^{2-3/2p'}\}\|V'\|_{2p} + I',
\end{aligned} \quad (3.5)$$

where

$$p^{-1} + p'^{-1} = 1, \qquad I = \int_0^\infty y^2(1+y^2)^{-2p'n}\,dy$$

and

$$I' = \|V'\phi_c(x)(2t)^{-3/2}h_0(x/2t)\|_2.$$

We have used the Hölder inequality in going from the first to the second line of (3.5). For $p < 3/2$, $p' > 1$ so that the first term of (3.5) is integrable on $[t_0, \infty)$. The second term can be taken care of by the use of a formula due to B Simon. Using the fact that

$$\|(2t)^{-3/2}h_0(x/2t)\|_2 = \|h_0\|_2 = \|h\|_2,$$
$$\|(2t)^{-3/2}h_0(x/2t)\|_\infty \le (2t)^{-3/2}\|h\|_1,$$

and the Riesz-Thorin[14] convexity theorem we obtain Simon's formula:

$$\|(2t)^{-3/2}h_0(x/2t)\|_{p'} \leqq K t^{-3(1-2/p')/2}\|h\|_p$$

for $p^{-1} + p'^{-1} = 1$, $1 \le p \le 2$. Thus with $s^{-1} + s'^{-1} = 1$ and using the Hölder inequality we have

$$\begin{aligned}
&\|V'\phi_c(x)(2t)^{-3/2}h_0(x/2t)\|_2 \\
&\le \|V'^2\|_{s'}^{1/2}\|(2t)^{-3/2}h_0(x/2t)\|_{2s} \\
&\le K'\|V'\|_{2s'} t^{-3/2s'}.
\end{aligned} \quad (3.6)$$

Equation (3.6) is integrable for $s' < 3/2$ as V' is assumed to have finite p norm for some $p < 3$.

Proof of Theorem 2.4: (a) From Theorem 2.1a we have that $W_\pm(H_1, H_D)$ are acc and cc. From Theorem 2.3 we have that $W_\pm(H_2, H_1)$ are acc. We prove the chain rule (valid although $H_D(t)$ is not a one-parameter unitary group):

$$W_\pm(H_2, H_1)\,W_\pm(H_1, H_D) = W_\pm(H_2, H_D). \quad (3.7)$$

Let $P_1(P_2)$ be the projection on the absolutely continuous spectrum subspace of $H_1(H_2)$. As $W_\pm(H_1, H_D)\mathcal{H} = P_1\mathcal{H}$ and $W_\pm(H_2, H_1)$ are acc then $W_\pm(H_2, H_D)\mathcal{H} = P_2\mathcal{H}$, i.e., $W_\pm(H_2, H_D)$ are acc. We establish (3.7) for $t \to +\infty$. We have $W_+(H_2, H_1)\,W_+(H_1, H_D) = $ s-lim $e^{itH_2}P_1 e^{-iH_D(t)}$ as $t \to \infty$. For $v \in L^2(R^3)$, we find

$$\begin{aligned}
&\|W_+(H_2, H_D)v - W_+(H_2, H_1)\,W_+(H_1, H_D)v\| \\
&= \|\lim_{t\to\infty}(e^{iH_2t}Ie^{-iH_D(t)}v - e^{iH_2t}P_1e^{-iH_D(t)}v)\| \\
&= \lim_{t\to\infty}\|(I - P_1)e^{-iH_D(t)}v\| \\
&= \lim_{t\to\infty}\|e^{iH_1t}(I - P_1)e^{-iH_D(t)}v\| \\
&= \|(I - P_1)W_+(H_1, H_D)v\|.
\end{aligned} \quad (3.8)$$

Equation (3.8) is zero since $W_+(H_1, H_D)$ is acc which implies $W_+(H_1, H_D) = P_1W_+(H_1, H_D)$.

(b) From Kodaira's theory of eigenfunction expansions,[4] the absolutely continuous spectrum subspace of H_2 corresponds with the continuous spectrum subspace that $W_\pm(H_2, H_D)$ are cc.

(c) In both (a) and (b), $W_\pm\mathcal{H} = P_2\mathcal{H}$ so that $S = W_+^* W_-$ is unitary.

Proof of Theorem 2.6: We first give two lemmas which will be used in the proof of part b.

Lemma 3.1. Let $u(r, k)$ be the solution of

$$[-d^2/dr^2 + l(l+1)/r^2 + V(r)]u = k^2u$$

used in the eigenfunction expansions of Sec. 1 with V satisfying the conditions of (1a)-(1c) such that

$$u(0, k) = 0$$

and

$$u(r,k) \simeq (2/\pi)^{1/2}\sin[kr - l\pi/2 + (\alpha/k)\ln 2kr + \delta(k)], \qquad r \to \infty.$$

Then $u(r,k)$ is continuous in $((r,k): r \geq 0, k > 0)$ and is bounded in the set $D' = (r,k): 0 \leq r \leq r_2$, $0 < k_1 \leq k \leq k_2)$ with the bound

$$|u(r,k)| \leq C r^{l+1} e^{r^\epsilon/\epsilon}, \tag{3.9}$$

where $V(r) \sim O(1/r^{2-\epsilon})$ as $r \to 0$.

Proof of Lemma 3.1: Following Sec. 22.25 of Titchmarsh[15] we solve the integral equation

$$y(r,k) = k^{-l-1/2} r^{1/2} J_{l+1/2}(kr) - (\pi/2) \times \int_0^\infty [J_{l+1/2}(kr) Y_{l+1/2}(ks) - J_{l+1/2}(kr) \times Y_{l+1/2}(ks)] r^{1/2} s^{1/2} V'(s) y(s,k) ds \tag{3.10}$$

by iteration with $V'(r) = \alpha/r + V(r)$, and $y(r,k)$ is the solution which behaves as r^{l+1} as $r \to 0$. We easily find $y(r,k) \leq C' r^{l+1} e^{r^\epsilon/\epsilon}$ in D'. From Theorem 5.3 of Kodaira[4]

$$u(r,k) = (2/\pi)^{1/2} \cdot |k/A(k)|\, y(r,k),$$

where $A(k)$ is continuous in $[k_1, k_2]$ and nonzero so the theorem follows.

Lemma 3.2: Let V satisfy (1a), (1b), and (2.1) and have compact support.

Let

$$h(r, k^2 + i\sigma) = (R_{1, k^2+i\sigma} V u_2)(r,k),$$

where the resolvent operator $R_{1,k^2+i\sigma} = (H_1 - k^2 - i\sigma)^{-1}$ is an integral operator in $L^2(0,\infty)$ with kernel $G_1(r, r', k^2 + i\sigma)$ $(\sigma \neq 0)$. Then

$$\lim_{\sigma \to 0+} h(r, k^2 + i\sigma) = (R_{1,k^2} V u_2)(r,k)$$

uniformly in any $D = ((r,k): 0 < r_1 \leq r \leq r_2$, $0 < k_1 \leq k \leq k_2)$, where R_{1,k^2} is the integral operator with kernel $G_1(r, r', k^2)$.

Proof of Lemma 3.2: From Theorem 20.21 of Stone[10] the resolvent operator $(H_1 - \lambda)^{-1}$, $\mathrm{Im}\lambda \neq 0$ is an integral operator of the Carleman type, denoted by $G_1(r,s,\lambda)$, given by

$$G_1(r,s,\lambda) = \begin{cases} u_1(r,\tau) w_1(s,\tau)/W(w_1,u_1), & 0 < r < s \\ w_1(r,\tau) u_1(s,\tau)/W(w_1,u_1), & 0 < s < r \end{cases}, \tag{3.11}$$

where $u_1(r,\tau)$ and $w_1(r,\tau)$ are independent solutions of $(L_1 - \lambda) y = 0$, $\tau = (\lambda)^{1/2}$, and $W(w_1, u_1)$ is the Wronskian. In Messiah's[1] notation, two linearly independent solutions are

$$F_l(\alpha/\tau, \tau r) = c_l e^{i\tau r} (\tau r)^{l+1} \times F(l + 1 + i(\alpha/\tau) | 2l + 2 | -2i\tau r),$$

$$G_l(\alpha/\tau, \tau r) = v_1(r, \tau) = c_l e^{i\tau r} (\tau r)^{l+1} G(l + 1 + i\alpha/\tau | 2l + 2 | -2i\tau r) \tag{3.12}$$

with $\mathrm{Re}\tau > 0$. F and G are confluent hypergeometric functions and

$$c_l(1/\tau) = [2^l/2l+1)!] \exp[-i\pi\alpha/\tau] | \Gamma[l + 1 = (i\alpha/\tau)|.$$

Since $F(\beta|\gamma|\rho)$ is entire in β and ρ, and $G(\beta|\gamma|\rho)$ is entire in β and analytic in ρ except for $-\infty < \rho \leq 0$ (see Lebedev,[16] pages 261-65), F_l and G_l are C^∞ in the three variables (r,k,σ) where $r \in [0,\infty)$ for F_l, $r \in (0,\infty)$ for G_l and $k \in (0,\infty)$, where we set $\lambda = k^2 + i\sigma$, $\mathrm{Re}\tau > 0$. For $\sigma > 0$ we have

$$w_1(r,\tau) = u^+(\alpha/\tau, \tau r) = [G_l(\alpha/\tau, \tau r) + i F_l(\alpha/\tau, \tau r)] \times [\arg\Gamma(l + 1 + (i\alpha/\tau))]$$

$$u_1(r,\tau) = (2/\pi)^{1/2} F_l(\alpha/\tau, \tau r); W^{-1}(w_1, u_1) = (\pi/2)^{1/2}(1/\tau)[\arg\Gamma(l + 1 + i\alpha/\tau)]. \tag{3.13}$$

The above considerations show that

$$\lim_{\sigma \to 0^+} u_1[r, (k^2 + i\sigma)^{1/2}] = u_1(r,k)$$

uniformly in any domain $0 < k_1 \leq k \leq k_2$, $0 \leq r \leq r_2$ and that

$$\lim_{\sigma \to 0^+} w_1(r, (k^2 + i\sigma)^{1/2}) = w_1(r,k)$$

uniformly in any domain $0 < k_1 \leq k \leq k_2$, $0 < r_1 \leq r \leq r_2$. Thus

$$\lim_{\sigma \to 0^+} G_1(r, s, k^2 + i\sigma) = G_1(r, s, k^2)$$

uniformly in any domain $0 < r_1 \leq r \leq r_2$, $0 \leq s \leq s_2$, $0 < k_1 \leq k \leq k_2$. Let $\mathrm{supp} V \subset [0, s_2]$. We have

$$\int_0^\infty |V(s) u_2(s,k)| ds \leq C' \int_0^{s_2} s^2 V^2(s) ds = C'' < \infty \tag{3.14}$$

so that

$$V(r) u_2(r,k) \in L^2(0,\infty) \text{ and } V(r) u_2(r,k) \in L^1(0,\infty)$$

with a bound C'' which is uniform in $0 < k_1 \leq k \leq k_2$. We find

$$\sup_D |h(r, (k^2 + i\sigma)^{1/2}) - h(r,k)| \leq \sup_D \int_0^{S_2} |G_1(r,s,k^2 + i\sigma) - G_1(r,s,k^2)| |V(s) u_2(r,k)| ds \leq C'' \sup_D |G_1(r,s,k^2 + i\sigma) - G_1(r,s,k^2)|,$$

which suffices to establish the lemma.

Proof of Theorem 2.6(a): This is obvious.

Proof of Theorem 2.6(b): As this proof is rather lengthy we give a brief outline. In Part I we prove the theorem for a continuous, cutoff V, denoted by V_n, with supp $V_n \subset [0, n+1]$. The equality $W_\pm(H_1 + V_n, H_1) = U_\pm(H_1 + V_n, H_1)$ is established following a method of Ikebe[17] and uses previous results on the existence and cc of the wave operators $W_\pm(H_2, H_1)$.[11] In Part II, following Kuroda[8], a limiting procedure is used which allows us to prove the result for V.

Part I: We will prove

$$W_\pm(H_1 + V_n, H_1) = F_2^* e^{\pm i(\delta_1 - \delta_2)} F_1 = U_\pm(H_1 + V_n, H_1)$$

(We suppress the n dependence of F_2^* and δ_2) for a continuous cut-off potential defined by

$$V_n(r) = \begin{cases} V(r), & 0 < r \leq n \\ [\mathrm{sgn}\, V(n)] \min(|V(r)|, |V(n)|(n + 1 - r)), & n \leq r \leq n+1 \\ 0, & n + 1 \leq r < \infty. \end{cases}$$

With this definition, $|V_n(r)| \le |V(r)|$. Throughout Part I we let $f \in C_0^\infty(0,\infty)'$ and g is such that $F_1 g \in C_0^\infty(0,\infty)'$, where $C_0^\infty(0,\infty)'$ is the set of all C^∞ functions with compact support contained in $(0,\infty)$. Furthermore we consider only the $t \to -\infty$ case, the proof for $t \to +\infty$ is similar. With f and g as above we follow Ikebe[17] (see Sec. 11) to establish that $U_\pm(H_1 + V_n, H_1)$ are well defined. We have

$$\begin{aligned}(U_- g)(r) &= \int_0^\infty u_2(r,k)\, e^{i(\delta_1-\delta_2)}(F_1 g)(k)\,dk \\ &= F_2^* e^{i(\delta_2-\delta_1)} F_1 g, \qquad (3.15)\\ (U_-^* f)(r) &= \int_0^\infty u_1(r,k)\, e^{i(\delta_1-\delta_2)}(F_2 f)(k)\,dk \\ &= F_1^* e^{i(\delta_1-\delta_2)} F_2 f,\end{aligned}$$

which can be extended to all $f, g \in L^2(0,\infty)$ by writing l.i.m. in front of the integrals. Note that

$$U_-(L^2(0,\infty)) \subset P_2(L^2(0,\infty)), \qquad (3.15'a)$$

$$U_-^*(L^2(0,\infty)) \subset P_1(L^2(0,\infty)). \qquad (3.15'b)$$

We also have that

$$U_-^* e^{iH_2 t} = e^{iH_1 t} U_-^*, \qquad U_-^* H_2 \subset H_1 U_-^*. \qquad (3.16)$$

We will now derive

$$U_-^*(H_1 + V_n, H_1)\, W_-(H_1 + V_n, H_1) = P_1.$$

Starting from

$$U_-^* W_- = \operatorname*{s-lim}_{t\to-\infty} U_-^* e^{i(H_1+V_n)t} e^{-tH_1} P_1 \qquad (3.17)$$

and, using the existence of the strong limit, the fact that $u_2 V_n \in L^2(0,\infty)$ and that $P_1 F_1 = F_1$, we arrive at

$$(U_-^* W_- f, P_1 g) = (U_-^* f, P_1 g) + \lim_{\delta\to 0^+} \int_0^\infty (f, R_{1,k^2+i\sigma} V_n u_2) \times (F_1 g)(k)\, e^{i(\delta_2-\delta_1)}\,dk. \qquad (3.18)$$

By Lemma 3.2 we can pass the limit inside the integral and inner product of (3.18) obtaining

$$\begin{aligned}(U_-^* W_- f, g) &= (U_-^* W_- f, P_1 g) \\ &= \int_0^\infty e^{-i(\delta_1-\delta_2)}(F_1 g)(k) \int_0^\infty \bar{f}(r) \times \phi(r,k)\,dr\,dk \qquad (3.19)\end{aligned}$$

with

$$\phi(r,k) = u_2(r,k) + \int_0^{n+1} G_1(r,r',k^2)\, V_n(r') \times u_2(r',k)\,dr', \qquad k \ne 0,$$

where $G_1(r,r',k^2)$ is given by (3.11). For fixed k ($k \ne 0$) we see that $G_1(r,r',k^2)$ is a bounded continuous function in a neighborhood of the r, r' origin.

Thus $\phi(r,k)$ is continuous in $0 \le r < \infty$ and bounded at the origin. Furthermore, $(L_1 - k^2)\phi(r,k) = 0$ so $\phi(r,k) = c_k u_1(r,k)$. Using the explicit form of $G_1(r,r',k^2)$ [see (3.11) and (3.13)], we have

$$\phi(r,k) = (u_2 + K w_1)(r,k) \qquad \text{for } r \ge n+1.$$

Thus, comparing the e^{ikr} part of the asymptotic form of $u_2(r,k)$ and $u_1(r,k)$ for $r \to \infty$, we find $c_k = e^{i(\delta_2-\delta_1)}$. Equation (3.19) becomes $(U_-^* W_- f, P_1 g = \int_0^\infty (F_1 g)(k) \int_0^\infty \bar{f}(r) u_1(r,k)\,dr\,dk = (P_1 f, g)$. By (3.15b),

$$P_1 U_-^* = P_1 \quad \text{so} \quad (U_-^* W_- f, g) = (P_1 f, g)$$

for arbitrary f and g dense in $L^2(0,\infty)$. Thus we have the $L^2(0,\infty)$ relation

$$U_-^* W_- = P_1. \qquad (3.20)$$

Using (3.20), we see that it is easy to show $U_- = W_-$. From Theorem 2.3 it follows that W_- is acc, so that $W_- W_-^* = P_2$. Thus

$$U_-^* W_- W_-^* = P_1 W_-^* = U_-^* P_2.$$

Taking adjoints we have $P_2 U_- = W_- P_1$. Using (3.15a) we obtain $U_- = W_-$. We have established on $L^2(0,\infty)$ that

$$W_\pm(H_1 + V_n, H_1) = F_2^{n*} e^{\pm i(\delta_1-\delta_2^n)} F_1. \qquad (3.21)$$

In (3.21) we have resurrected the n dependence of F_2^* and δ_2. In what follows F_2 and δ_2 will be associated with $H_2 = H_1 + V$.

Part II: We will follow a method of Kuroda[8] which allows passage to the limit $n \to \infty$ in (3.21), obtaining $U_-(H_2, H_1) = W_-(H_2, H_1)$. We will prove the following:

$$\operatorname*{s-lim}_{n\to\infty} W_\pm(H_1 + V_n, H_1) = W_\pm(H_2, H_1), \qquad (3.22)$$

$$\lim_{n\to\infty} \delta_2^n(k) = \delta_2(k) \ \ [\text{uniformly in } (0,\infty)], \qquad (3.23)$$

$$\operatorname*{s-lim}_{n\to\infty} F_2^* e^{\pm i(\delta_1-\delta_2^n)} F_1 = F_2^* e^{\pm i(\delta_1-\delta_2)} F_1. \qquad (3.24)$$

From (3.21), (3.22), and (3.24) part b of the theorem follows.

Proof of (3.22): By Theorem 2 of Kuroda[12] it is sufficient to show that $|V - V_n|^{1/2}(H_1-\lambda)^{-1}$ is Hilbert-Schmidt and

$$\lim_{n\to\infty} \| |V - V_n|^{1/2}(H_1-\lambda)^{-1}\|_{H\text{-}S} = 0$$

for some λ (which we take to be real) in the resolvent set of H_1. Since $|(V - V_n)(r)| \le 2|V(r)|$ and

$$\| |V|^{1/2}(H_1-\lambda)^{-1}\|_{H\text{-}S} < \infty,$$

we have

$$\| |V - V_n|^{1/2}(H_1-\lambda)^{-1}\|_{H\text{-}S}^2 \le 2\| |V|^{1/2}(H_1-\lambda)\|_{H\text{-}S}^2 < \infty.$$

The Lebesgue dominated convergence theorem gives

$$\lim_{n\to\infty} \| |V - V_n|^{1/2}(H_1-\lambda)^{-11}\|_{H\text{-}S}^2 = 0 \text{ so that } \operatorname*{s-lim}_{n\to\infty} W_\pm(H_1 + V_n, H_1) = W_\pm(H_2, H_1).$$

Proof of (3.23): Let $u_2^n(r,k)$ satisfy $(L_1 + V_n - k^2)u_2^n = 0$ with the boundary and asymptotic conditions

$\lim_{r\to 0} u_2^n(r,k) = 0, \quad u_2(r,k) \sim (2/\pi)^{1/2} \sin[kr - (l\pi/2) + (\alpha/k)\ln 2(kr) - \delta_2^n(k)]$

as $r \to \infty$. Let $\tilde{u}_2^n(r,k)$ also satisfy

$(L_1 + V_n - k^2)\tilde{u}_2^n = 0$ and $\tilde{u}_2^n(r,k) = u_2(r,k)$ for $0 \leq r \leq n$.

We find for $r \geq n+1$

$$(\tilde{u}_2^n - u_2)(r,k) = W(u_1,v_1)^{-1}\left(\int_n^{n+1} K_1(r,s,k)[V(s)u_2(s,k) - V_n(s)u_2^n(s,k)]ds + \int_{n+1}^r K_1(r,s,k)V(s)u_2(s,k)ds\right) \quad (3.25)$$

with $K_1(r,s,k) = u_1(r,k)v_1(s,k) - u_1(s,k)v_1(r,k)$.

Kuroda[8] derives a formula analogous to (3.25) for the case $\alpha = 0$ which is correct only for $l = 0$. However, using the appropriate kernel for $l \neq 0$, his argument goes through. Since $V_n(r) = V(r)$ for $0 \leq r \leq n$, we readily obtain

$$\tilde{u}_2^n(r,k) = c_{nk}u_2^n(r,k), \quad (3.26)$$

where the c_{nk} can be chosen positive.

Starting from (3.25) we derive

$$\lim_{n\to\infty} (\tilde{u}_2^n - u_2)(r,k) = 0 \text{ uniformly in } 0 \leq r < \infty \text{ and } 0 < k_1 \leq k \leq k_2.$$

Hence, by virtue of the asymptotic forms of $\tilde{u}_n^2$ and u_2, and the periodic property of the sine function, we get

$$\lim_{n\to\infty} \{[\sin[kr - \tfrac{1}{2}l\pi + (\alpha/k)\ln 2kr - \delta_2] - c_{nk}\sin[kr - \tfrac{1}{2}l\pi + (\alpha/k)\ln 2kr - \delta_2^n]\} = 0$$

uniformly for all r and $0 < k_1 \leq k \leq k_2$. This readily implies

$$\lim_{n\to\infty} \delta_2^n(k) = \delta(k), \quad \lim_{n\to\infty} c_{nk} = 1 \quad (3.27)$$

uniformly for $0 < k_1 \leq k \leq k_2$. Then (3.26), (3.27) also show that

$$\lim_{n\to\infty} u_2^n(r,k) = u_2(r,k) \quad (3.28)$$

uniformly in r and in $0 < k_1 \leq k \leq k_2$.

Proof of (3.24): Kuroda[8] states that (3.24) can be proved (when $\alpha = 0$) using a standard argument. We have not been able to construct his argument, so we prove (3.24) directly. Let

$$g = F_1 f, \quad g \in C_0^\infty(0,\infty)'$$

and set (for the $t \to -\infty$ case)

$$h_n(r) = (F_2^{n*}e^{(\delta_2^n - \delta_1)}F_1 f)(r) = \int_0^\infty u_2^n(r,k)\,e^{(\delta_2^n - \delta_1)}g(k)\,dk.$$

From (3.27) and (3.28), we have

$$\lim_{n\to\infty} h_n(r) = h(r) = \int_0^\infty u_2(r,k)\,e^{i(\delta_2 - \delta_1)}g(k)\,dk$$

uniformly in $r \in [0,\infty)$. Equations (3.21) and (3.22) guarantee the existence of an h' such that

$$\lim_{n\to\infty} \|h_n - h'\| = 0.$$

Equation (3.24) is readily established by showing that $h = h'$.

Proof of Theorem 2.6(c): We first show that

$$W_\pm(H_2,H_D) = W_\pm(H_2,H_1)W_\pm(H_1,H_D) = F_2^* e^{\pm i(\delta_1 - \delta_2)}F_1F_1^* e^{\mp i\delta_1}F_0 = F_2^* e^{\mp i\delta_2}F_0. \quad (3.29)$$

To show (3.29), it is sufficient to show that $P_1' = F_1F_1^*$ acts as the identity in $L^2(0,\infty)$. This follows since

$$(u, W_\pm^*(H_1,H_D)W_\pm(H_1,H_D)v) = (u,v) = (e^{\mp i\delta_1}F_0u, e^{\mp i\delta_1}F_0v) = (e^{\mp i\delta_1}F_0u, F_1F_1^* e^{\mp i\delta_1}F_0v)$$

and $e^{\mp i\delta_1}F_0$ is onto $L^2(0,\infty)$. To show that

$$S = W_+^*(H_2,H_D)W_-(H_2,H_D) = F_0^* e^{i\delta_2}F_2F_2^* e^{i\delta_2}F_0 = F_0^* e^{i2\delta_2}F_0, \quad (3.30)$$

it is sufficient to show that $F_2F_2^*$ acts as the identity in $L^2(0,\infty)$ in (3.30). This follows from (3.29) and the fact that $e^{\mp i\delta_2}F_0$ is onto $L^2(0,\infty)$ since

$$(u, W_\pm^*(H_2,H_D)W_\pm(H_2,H_D)v) = (u,v) = (e^{\mp i\delta_2}F_0u, e^{\mp i\delta_2}F_0v) = (e^{\mp i\delta_2}F_0u, F_2F_2^* e^{\mp i\delta_2}F_0v).$$

APPENDIX A

We give formal expressions in three- and one-dimensional form corresponding to those used in the text. A generalized Fourier transform in three dimensions we write as

$$f_i(q) = \int \overline{\Phi}_i(x,q)f(x)dx, \quad i = 0,1,2. \quad (A1)$$

Equation (A1) gives a mapping from $\mathcal{L}^2(\mathcal{R}^3)$ onto the continuous spectrum subspace of H_i. In addition to the kernel $\Phi_i(x,q) \equiv \Phi_i^-(x,q)$, we also have the related kernel $\overline{\Phi}_i(x,-q) \equiv \Phi_i^+(x,q)$. These kernels have the expansions

$$\Phi_i^\pm(x,q) = (kr)^{-1} \times \sum_{\substack{l=0\\|m|<l}}^{\infty} i^l e^{\mp i\delta_i^l(k)}u_i^l(r,k)\overline{Y}_{lm}(\hat{q})Y_{lm}(\hat{x}). \quad (A2)$$

The radial functions $u_i(r,k)$ have a phase shifted e^{ikr} asymptotic part for $i = 1,2$; for $i = 0$, $\delta_0^l(k) = 0$ for all l, $\delta_1^l(k) = \arg\Gamma[l + 1 + (i\alpha/k)]$ with $\alpha = e_1e_2m$. $\Phi_0^\pm(x,q) = (2\pi)^{-3/2}e^{iq\cdot x}$, the kernel of the $\mathcal{L}^2(\mathcal{R}^3)$ Fourier transform, and

$$\Phi_1^-(x,q) = (2\pi)^{-3/2} e^{iq\cdot x} e^{-\pi\alpha/2k} \times \Gamma(1+i\alpha/k)\, F(-i\alpha/k|1|ikr - iq\cdot x), \quad \text{(A3)}$$

where F is the confluent hypergeometric function. The asymptotic form of $\Phi_1^-(x,q)$ is

$$\Phi_1^-(x,q) \sim \Phi_{1_{\rm inc}}(x,q) + \Phi_{1_{\rm sc}}(x,q), \qquad |r - \hat{q}\cdot x| \to \infty, \quad \text{(A4a)}$$

$$\Phi_{1_{\rm inc}}^-(x,q) = (2\pi)^{-3/2} \exp\{i[q\cdot x + \alpha \log(kr - q\cdot x)/k]\} \times [1 + (\alpha/ik^2)(kr - q\cdot x) + \cdots], \quad \text{(A4b)}$$

$$\begin{aligned}\Phi_{1_{\rm sc}}^-(x,q) &= (2\pi)^{-3/2}(-\alpha/k)(kr - q\cdot x)^{-1}\,\Gamma(1+i\alpha/k) \\ &\quad\times (\Gamma(1-i\alpha/k))^{-1} \exp\{i[kr - \alpha \log(kr - q\cdot x)/k]\} \\ &= (2\pi)^{-3/2} r^{-1} \exp\{i[kr - \alpha\log(2kr)/k]\} f_{\rm ck},\end{aligned} \quad \text{(A4c)}$$

where $\cos\theta = \hat{q}\cdot\hat{x}$, $\hat{x} = \hat{q}'$, and

$$\begin{aligned}f_{\rm ck}(\theta) &= (-\alpha/k)[2k\sin^2(\theta/2)]^{-1}\exp\{-i(\alpha/k) \\ &\quad\times \log[\sin^2(\theta/2)] + 2i\delta_1^0\} \\ &= (-\alpha/k)(2k)^{-1}[(1-\cos\theta)/2]^{-1-i\alpha/k} \\ &\quad\times \exp(2i\delta_1^0)\end{aligned} \quad \text{(A4d)}$$

is identified as the Coulomb scattering amplitude.

As sesquilinear forms, the $\Phi_i^\pm(x,q)$ obey

$$\int \overline{\Phi}_i^\pm(x,q')\,\Phi_i^\pm(x,q)\,dx = \delta(q'-q) + k^{-2}\delta(k'-k) \sum_{\substack{l=0\\|m|<l}}^{\infty} Y_{lm}(\hat{q})\, Y_{lm}(\hat{q}'). \quad \text{(A5)}$$

The time-independent operators of Eq. (2.2) in three-dimensional form are

$$(U_\pm(H_1,H_D)f)(x) = \int \Phi_1^\pm(x,q)\left(\int \overline{\Phi}_0(x',q) f(x')\,dx'\right) dq, \quad \text{(A6)}$$

$$(U_\pm^*(H_1,H_D)f)(x) = \int \Phi_0(x,q)\left(\int \overline{\Phi}_1^\pm(x',q) f(x')\,dx'\right) dq. \quad \text{(A7)}$$

Also in (2.3) we have

$$(U_\pm(H_2,H_1)f)(x) = \int \Phi_2^\pm(x,q)\left(\int \overline{\Phi}_1^\pm(x',q) f(x')\,dx'\right) dq \quad \text{(A8)}$$

$$(U_\pm^*(H_2,H_1)f)(x) = \int \Phi_1^\pm(x,q)\left(\int \Phi_2^\pm(x',q) f(x')\,dx'\right) dq. \quad \text{(A9)}$$

Then with $U_\pm(H_2,H_D) = U_\pm(H_2,H_1)\,U_\pm(H_1,H_D)$, we have

$$(U_\pm(H_2,H_D)f)(x) = \int \Phi_2^\pm(x,q) f_0(q)\,dq. \quad \text{(A10)}$$

The S' operator of Theorem 1.6 as a sesquilinear form in three-dimensions is

$$\begin{aligned}(g,S'f) &= (U_+(H_2,H_D)g,\, U_-(H_2,H_D)f) \\ &= \int \overline{g}_0(q')\left(\int \overline{\Phi}_2^+(x,q')\,\Phi_2^-(x,q)\,dx\right) f_0(q)\,dq\,dq'.\end{aligned} \quad \text{(A11)}$$

Taking into account (A2) we have the one—dimensional form of (A6)–(A10) where we will suppress the l dependence:

$$\begin{aligned}(U_\pm(H_1,H_D)f)(r) &= \int_0^\infty e^{\mp i\delta_1(k)} u_1(r,k)(F_0 f)(k)\,dk \\ &\equiv (F_1^* e^{\mp i\delta_1} F_0 f)(r),\end{aligned} \quad \text{(A6}')$$

$$(U_\pm^*(H_1,H_D)f)(r) = (F_0^* e^{\pm i\delta_1} F_1 f)(r), \quad \text{(A7}')$$

$$(U_\pm(H_2,H_1)f)(r) = (F_2^* e^{\mp i(\delta_2-\delta_1)} F_1 f)(r), \quad \text{(A8}')$$

$$(U_\pm^*(H_2,H_1)f)(r) = (F_1^* e^{\pm i(\delta_2-\delta_1)} F_2 f)(r), \quad \text{(A9}')$$

$$(U_\pm(H_2,H_D)f)(r) = (F_2^* e^{\mp i\delta_2} F_0 f)(r). \quad \text{(A10}')$$

Now returning to (A11) we substitute the expansions of (A2) to obtain

$$(g,S'f) = \int \overline{g}_0(q')\,k^{-2}\delta(k'-k) \times \sum_{\substack{l=0\\|m|\le l}}^{\infty} [e^{2i\delta_2^l(k)}\,\overline{Y}_{lm}(\hat{q})\,Y_{lm}(\hat{q}')] f_0(q)\,dq\,dq'. \quad \text{(A12)}$$

Using (A5) and $\delta(E'-E) = (2k)^{-1}\delta(k'-k)$, we then have

$$(g,(S'-I)f) = \int \overline{g}_0(q')[-2\pi i\,\delta(E'-E) \times T(q',q;k=k')] f_0(q)\,dq\,dq', \quad \text{(A13)}$$

where the scattering amplitude $f_k(\hat{q}\cdot\hat{q}')$ is given by

$$f_k(\hat{q}\cdot\hat{q}') = -2\pi^2\, T(q',q;k=k') = (2ik)^{-1}\sum_{l=0}^{\infty}(2l+1) \times \{\exp[2i\delta_2^l(k)] - 1\}\, P_l(\hat{q}\cdot\hat{q}'). \quad \text{(A14)}$$

The scattering amplitude of (A14) makes up part of a kernel of a sesquilinear form and, as such, is well defined. However, from a physical point of view, we are interested in its properties as a function of k and $\hat{q}\cdot\hat{q}'$. Unfortunately we have not been able to rigorously show that (A14), in the case $V = 0$, agrees with (A4d).

APPENDIX B

We give a nonrigorous argument that indicates that the wave operators $W_\pm(H_j,H_i)$ $(i,j = 1,2)$ exist and are absolutely continuous complete under the assumption that V is spherically symmetric and satisfies

$$|V| \le K\,(r^{-2+\epsilon} + r^{-1-\epsilon}), \qquad \epsilon > 0.$$

We use a technique of Kuroda[12] which he successfully used for the case $V = 0$. By Kuroda's theorem for forms to show absolutely continuous completeness and existence of the wave operators, it is sufficient to show that $|V|^{1/2}(H_1 - \lambda)^{1/2}$ is Hilbert–Schmidt for some $\lambda < \inf(\text{spectrum of } H_1)$. Using the representation

$$\begin{aligned}&[(H_1-\lambda)^{1/2} f](r) \\ &= \lim_{n\to\infty}\left(\sum_{m=0}^{n} u_{lm}(r)(u_{lm},f)(\lambda_m - \lambda)^{-1/2} + \int_{n-1}^{n} u_1(r,k) \times (F_1 f)(k)\,(k^2-\lambda)^{-1/2}\,dk\right), \\ &\qquad f \in C_0^\infty(0,\infty),\end{aligned}$$

we have $\||V|^{1/2}(H_1-\lambda)^{-1/2}\|_{H\text{-}S}^2 = S + I$, where

$$S = \sum_{n=0}^{\infty}(u_{ln}, |V| u_{ln})(\lambda_n - \lambda)^{-1},$$

$$I = \int_0^\infty \int_0^\infty |V(r)| u_1^2(r,k)(k^2-\lambda)^{-1} dk dr.$$

Choosing λ such that $\mathrm{supp}(\lambda_n - \lambda) < 1$, we find the bound

$$S \leq \sum_n [(u_{ln}, r^{-2} u_{ln}) + (u_{ln}, r^{-1} u_{ln})]$$

$$\leq \sum_n [(2l+1)^{-1} n^{-3} + n^{-2}] < \infty.$$

Using the more natural variables $p = kr$, we have

$$I = \int_0^\infty \int_0^\infty |V(p/k)| u_1^2(p,k)(k^2-\lambda)^{-1} k^{-1} dk dp,$$

where

$$|V(p/k)| \leq K (k^{2-\epsilon} p^{-2+\epsilon} + k^{1+\epsilon} p^{-1-\epsilon}),$$

$$|V(p/k)|(k^2-\lambda) k^{-1}$$
$$\leq K' k^\epsilon (1+k)^{-1-2\epsilon} (1+p)^{1-2\epsilon} p^{\epsilon-2}.$$

Recalling that

$$u_1(p,k) \propto c_l e^{ip} p^{l+1} F(l+1+i\alpha/k | 2l+2 | -2ip),$$

where

$$c_l \propto e^{-\pi\alpha/2k} |\Gamma(l+1+i\alpha/k)|$$

$$\propto [(\alpha/k) e^{-\pi\alpha/k} \sinh^{-1}(\alpha/k)] \Pi_{m=1}^{l} (m^2 + \alpha^2/k^2),$$

we find that

$$c_l = \begin{cases} k^{-l-1/2} e^{-2\pi\alpha/k} \to 0, & k \to 0,\ \alpha/k > 0, \\ k^{-1/2}, & k \to 0,\ \alpha/k < 0, \\ 1, & k \to \infty. \end{cases}$$

For $p \to 0$, $k \to \infty$, $F(l+1+i\alpha/k | 2l+2 | -2ip) \to 1$; for $p \to \infty$, $k \to \infty$,

$$u_1(p,k) \sim (2/\pi)^{1/2} \sin[p - (\alpha/k)\log(2p) - \tfrac{1}{2} l\pi + \delta_1^l)]$$
$$\to (2/\pi)^{1/2} \sin(p - \tfrac{1}{2} l\pi,\ p \geq l(l+1) + (\alpha/k).$$

From Landau and Lifschitz[18] (see p. 150) we have for $k \to 0$,

$$u_1(r,0) k^{-1/2} \propto (2r)^{1/2} J_{2l+1}[(8r)^{1/2}].$$

These considerations indicate that

$$u_1(p,k) \leq K (1+k)^{1/2} k^{-1/2} (1+p)^{-l-1} p^{l+1}.$$

If the above bound for u_1 holds, then $\mathcal{I} < \infty$.

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On Weyl and Lyra Manifolds*

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It is shown that Weyl's geometry and an apparently similar geometry suggested by Lyra are special cases of manifolds with more general connections. The difference between the two geometries and their relationship with Riemannian geometry are clarified by giving a global formulation of Lyra's geometry. Finally the outline of a field theory based on the latter geometry is given.

1. INTRODUCTION

Shortly after Einstein's general theory of relativity Weyl[1] suggested the first so-called unified field theory based on a generalization of Riemannian geometry. In retrospect, it would seem more appropriate to call Weyl's theory a geometrized theory of gravitation and electromagnetism (just as the general theory was a geometrized theory of gravitation only), rather than a unified field theory. It is not quite clear to what extent the two fields have been unified, even though they acquire (different) geometrical significances in the same geometry. The theory was never taken seriously because it was based on the concept of nonintegrability of length transfer, and, as pointed out by Einstein, this implies that spectral frequencies of atoms depend on their past histories and therefore have no absolute significance. Nevertheless, Weyl's geometry provides an interesting example of non-Riemannian connections, and recently Folland[2] has given a global formulation of Weyl manifolds thereby clarifying considerably many of Weyl's basic ideas.

In 1951 Lyra[3] suggested a modification of Riemannian geometry which bears a remarkable resemblance to Weyl's geometry. But in Lyra's geometry, unlike Weyl's, the connection is metric preserving as in Riemannian geometry; in other words, length transfers are integrable. Lyra also introduced the notion of a gauge and in the "normal" gauge the curvature scalar is identical to that of Weyl. It is thus possible[4] to construct a geometrized theory of gravitation and electromagnetism much along the lines of Weyl's "unified" field theory without, however, the inconvenience of nonintegrable length transfer.